On Certain Properties of Symmetric, Skew Symmetric, and Orthogonal Matrices.

BY W. H. METZLER.

§1.—INTRODUCTION.

In the Proceedings of the London Mathematical Society (Vol. XXII, Nos. 427, 428), Dr. Henry Taber has proved the following theorem: A real symmetric matrix less one of its multiple latent roots has a nullity equal to its vacuity. He also states and proves that this same property is characteristic of real skew symmetric and of real orthogonal matrices, that the latent roots of a real skew symmetric matrix are either zero or pure imaginary, and that the modulus of every latent root of a real orthogonal matrix is equal to unity. The method employed by Dr. Taber is an extension of that given by A. Buchheim in the Mess. Math. ((2), Vol. XIV). By employing a different method I shall prove the same properties of these matrices.

In addition, I shall give a representation of an orthogonal matrix which admits of both $+1$ and -1 as latent roots, and which at the same time contains a number of arbitrary parameters.

In what follows I shall consider only real matrices, so that when speaking of a matrix it will be understood that a real matrix is meant.

§2.—*Symmetric Matrices.*

It is a well-known theorem that the latent roots of a symmetric matrix are all real. Suppose the symmetric matrix ϕ has as latent roots $g_1, g_2, \ldots . g_s$ occurring $p_1, p_2, \ldots . p_s$ times respectively, then it is also well known that every first minor of the content of ϕ has as roots $g_1, g_2, \ldots . g_s$ occurring $p_1-1, p_2-1, \ldots . p_s-1$ times respectively, and generally every λ^{th} minor of the content of ϕ has as

roots $g_1, g_2, \ldots . g_s$ occurring $p_1 - \lambda$, $p_2 - \lambda$, $\ldots . p_s - \lambda$ times respectively.* The latent roots of the matrix $(\phi - g_1)$ will then, according to the law of latency, be $0, g_2 - g_1, \ldots . g_s - g_1$ occurring $p_1, p_2, \ldots . p_s$ times respectively; that is, $(\phi - g_1)$ has a vacuity p_1. Obviously every minor of the content of $(\phi - g_1)$ up to the $(p_1 - 1)^{\text{st}}$ is vacuous and therefore $(\phi - g_1)$ has a nullity p_1.

§3.—*Skew Symmetric Matrices.*

The latent function of a skew symmetric matrix of even order may be written in the form

$$x^{2n} + \Sigma A_1^2 x^{2n-2} + \Sigma A_2^2 x^{2n-4} + \ldots . + \Sigma A_{n-1}^2 x^2 + A_n^2,$$

where the exponents are all even and the coefficients are all positive; consequently if $A_n^2 \mp 0$, the roots of this function are all imaginary. If $A_n^2 = 0$, $\Sigma A_{n-1}^2 = 0$, $\Sigma A_{n-2}^2 = 0 \ldots . . \Sigma A_{n-\lambda+1}^2 = 0$, but $\Sigma A_{n-\lambda} \mp 0$, then the latent function becomes

$$x^{2\lambda}\{x^{2n-2\lambda} + \Sigma A_1^2 x^{2n-2\lambda-2} + \ldots . + \Sigma A_{n-\lambda}^2\},$$

and therefore 2λ of the roots are zero and the remainder are imaginary. If the order of the matrix is odd, then there is at least one latent root zero since its determinant vanishes, and as in the case of even order, the roots are either zero or imaginary. But the square of a skew symmetric matrix is a symmetric matrix, and since the latent roots of the square of a matrix are the squares of the latent roots of the matrix, we see that the latent roots of the skew symmetric matrix must have been all pure imaginary to have their squares real. Consequently the latent roots of a skew symmetric matrix are either zero or pure imaginary.

Suppose the skew symmetric matrix ϕ has as latent roots

$g_1 i$	occurring	p_1	times,
$-g_1 i$	"	p_1	"
$g_2 i$	"	p_2	"
$-g_2 i$	"	p_2	"
etc.,		etc.	

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* Vide Burnside and Panton, Theory of Equations, 2d ed., art. 129, examples 32 and 33.

Then $(\phi - g_1 i)$ is a skew matrix whose latent roots are

$$\begin{array}{rccc} 0 & \text{occurring} & p_1 & \text{times,} \\ -2g_1 i & \text{"} & p_1 & \text{"} \\ (g_2 - g_1)\,i & \text{"} & p_2 & \text{"} \\ -(g_2 + g_1)\,i & \text{"} & p_2 & \text{"} \\ \text{etc.,} & & \text{etc.} & \end{array}$$

Also $(\phi^2 + g_1^2)$ is a symmetric matrix whose latent roots are

$$\begin{array}{rccc} 0 & \text{occurring} & 2p_1 & \text{times,} \\ -g_2^2 + g_1^2 & \text{"} & 2p_2 & \text{"} \\ \text{etc.,} & & \text{etc.} & \end{array}$$

Again, $N_y[\phi - g_1 i] \overline{\overline{<}} p_1$, where $N_y[\phi] \overline{\overline{<}} k$ means "ϕ has a nullity equal to or less than k"; $N_y[\phi^2 + g_1^2] = 2p_1$, $(\phi^2 + g_1^2)$ being a symmetric matrix, and therefore, as we have shown, has a nullity equal to its vacuity;

$$N_y[\phi + g_1 i] \overline{\overline{<}} p_1.$$

But $\phi^2 + g_1^2 = (\phi - g_1 i)(\phi + g_1 i)$,

and therefore $$N_y[\phi - g_1 i] = p_1,$$

and $$N_y[\phi + g_1 i] = p_1,$$

since the product has a nullity $2p_1$ and each of the factors a nullity at most equal to p_1. The vacuity of $(\phi - g_1 i)$ is p_1, and therefore a skew symmetric matrix less one of its latent roots has a nullity equal to its vacuity.

§4.—*Orthogonal Matrices.*

If ϕ is an orthogonal matrix, then the content of ϕ is equal to plus or minus unity and $\phi\breve{\phi} = \breve{\phi}\phi = 1$, where $\breve{\phi}$ is the transverse of ϕ;

$$\therefore\ \breve{\phi} = \phi^{-1}.$$

Let the latent roots of ϕ, which are the same as the latent roots of $\breve{\phi}$, be $g_1, g_2, \ldots g_s$ occurring $p_1, p_2, \ldots p_s$ times respectively. The roots of ϕ^{-1} will be $g_1^{-1}, g_2^{-1}, \ldots g_s^{-1}$ occurring $p_1, p_2, \ldots p_s$ times respectively. But the equa-

tion $\phi = \phi^{-1}$ shows that the latent roots of ϕ^{-1} are the same as the latent roots of $\breve{\phi}$, and consequently the same as the latent roots of ϕ.

Two possible cases present themselves,

$$1).\quad g_\lambda = g_\lambda^{-1} \text{ and } \therefore\ g_\lambda = \pm 1,$$
$$2).\quad g_\mu = g_\nu^{-1}.$$

If then g_1 is a latent root p_1 times, g_1^{-1} will also be a latent root p_1 times.

Since the product of all the latent roots of ϕ is equal to ± 1, we see that if the order of the matrix is odd, at least one of its latent roots must be ± 1. We may therefore take the roots of ϕ to be

$$\begin{array}{lcll} g_1 & \text{occurring} & p_1 & \text{times,} \\ g_1^{-1} & \text{"} & p_1 & \text{"} \\ g_2 & \text{"} & p_2 & \text{"} \\ g_2^{-1} & \text{"} & p_2 & \text{"} \\ \text{etc.,} & & \text{etc.,} & \end{array}$$

where g_λ may be equal to g_λ^{-1}.*

The latent roots of the symmetric matrix $(\phi + \breve{\phi})$ are

$$\begin{array}{lcll} r_1 = g_1 + g_1^{-1} & \text{occurring} & 2p_1 & \text{times,} \\ r_2 = g_2 + g_2^{-1} & \text{"} & 2p_2 & \text{"} \\ r_3 = g_3 + g_3^{-1} & \text{"} & 2p_3 & \text{"} \\ \quad\text{etc.,} & & \text{etc.,} & \end{array}$$

all of which are real.

The product $(\phi - g_1)(\breve{\phi} - g_1) = \phi\breve{\phi} - g_1(\phi + \breve{\phi}) + g_1^2 = 1 + g_1^2 - g_1(\phi + \breve{\phi})$ is a symmetric matrix whose latent roots are

$$\begin{array}{rcll} 0 & \text{occurring} & 2p_1 & \text{times,} \\ 1 + g_1^2 - g_1(g_2 + g_2^{-1}) & \text{"} & 2p_2 & \text{"} \\ \text{etc.,} & & \text{etc.} & \end{array}$$

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* That the orthogonal matrix ϕ is symmetric when its latent roots are all real (i. e. equal to ± 1) may be shown as follows:

The matrix $(\phi - \breve{\phi})$ is skew symmetric, having as latent roots $\frac{g_1^2 - 1}{g_1}$, $\frac{g_2^2 - 1}{g_2}$, etc., occurring $2p_1$, $2p_2$, etc., times respectively. But if the latent roots of ϕ are real, then all the latent roots of $(\phi - \breve{\phi})$ are zero. The sum of the products of the latent roots, two at a time, is equal to the sum of the principal minors of order two of the content of ϕ, and consequently we have $\Sigma\,(\phi_{rs} - \phi_{sr})^2 = 0$, where ϕ_{rs} is the constituent of ϕ in the r^{th} row and s^{th} column,

$$\therefore\ \phi_{rs} = \phi_{sr}:$$

that is, ϕ is symmetric.

Again,
$$N_y[\phi - g_1] \overline{\overline{<}} p_1,$$
$$N_y[\breve{\phi} - g_1] \overline{\overline{<}} p_1;$$
but
$$N_y[(\phi - g_1)(\breve{\phi} - g_1)] = 2p_1;^*$$
$$\therefore\ N_y[\phi - g_1] = N_y[\breve{\phi} - g_1] = p_1.$$

The vacuity of $(\phi - g_1)$ is p_1, and therefore an orthogonal matrix less one of its latent roots has a nullity equal to its vacuity.

The matrix $(\phi - \breve{\phi})$ is skew symmetric, having as latent roots $\frac{g_1^2 - 1}{g_1}$, $\frac{g_2^2 - 1}{g_2}$, etc., which are either zero or pure imaginary.

We have then
$$\frac{g_1^2 + 1}{g_1} = r_1,$$
$$\frac{g_1^2 - 1}{g_1} = h_1 i,$$
where h_1 is real, therefore
$$\frac{r_1}{2} + \frac{h_1}{2} i = g_1$$
and
$$\frac{r_1}{2} - \frac{h_1}{2} i = \frac{1}{g_1},$$
$\therefore\ \left(\frac{r_1}{2}\right)^2 + \left(\frac{h_1}{2}\right)^2 = 1$; that is, the modulus of g_1 is unity, and similarly the modulus of each of the g's is unity.

The Representation of an Orthogonal Matrix.

The representation of an orthogonal matrix in terms of $\frac{n(n-1)}{2}$ arbitrary quantities was given by M. Hermit as follows:

$$\phi = \left(\begin{array}{cccc} 2\frac{\beta_{11} - \Delta}{\Delta} & 2\frac{\beta_{12}}{\Delta} & 2\frac{\beta_{13}}{\Delta} & \cdots\cdots \\ 2\frac{\beta_{21}}{\Delta} & 2\frac{\beta_{22} - \Delta}{\Delta} & 2\frac{\beta_{23}}{\Delta} & \cdots\cdots \\ 2\frac{\beta_{31}}{\Delta} & 2\frac{\beta_{32}}{\Delta} & 2\frac{\beta_{33} - \Delta}{\Delta} & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right)^\dagger$$

*The matrix $1 + g_1^2 - g_1(\phi + \breve{\phi}) = g_1\{r_1 - (\phi + \breve{\phi})\} = \psi$ though symmetric, is not a real matrix unless g_1 is real. It is, however, obviously true that ψ less either of its latent roots has a nullity equal to its vacuity.

† Camb. and Dub. Math. Jour., Vol. IX (1858), p. 63; vide Salmon, Higher Algebra, art. 44.

where β_{11}, β_{12}, etc., are the minors of the arbitrary skew determinant

$$\Delta = \begin{vmatrix} 1 & b_{12} & b_{13} \cdots\cdots, \\ -b_{12} & 1 & b_{23} \cdots\cdots, \\ -b_{13} & -b_{23} & 1 \quad \cdots\cdots, \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

If ψ denotes the skew symmetric matrix

$$\begin{pmatrix} 0 & b_{12} & b_{13} \cdots\cdots, \\ -b_{12} & 0 & b_{23} \cdots\cdots, \\ -b_{13} & -b_{23} & 0 \quad \cdots\cdots, \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

then we easily find that

$$\phi = 2/\Delta\{\psi + \psi^2\} + 1 \text{ if } \phi \text{ is of } 3^{\text{d}} \text{ order,}$$
$$\phi = 2/\Delta\{\psi + \psi^2 + \psi^3 + \psi(\Delta - 1 - d^2) - d^2\} + 1, \text{ where}$$
$$d^2 = (b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23})^2 \text{ if } \phi \text{ is of } 4^{\text{th}} \text{ order,}$$

etc.

Another representation of an orthogonal matrix in terms of an arbitrary skew symmetric matrix is the following:

$$\phi = \frac{1+\psi}{1-\psi}.^*$$

I shall now show that these two representations are virtually the same. The matrix

$$(1-\psi) = \begin{pmatrix} 1 & -b_{12} & -b_{13} \cdots\cdots, \\ b_{12} & 1 & -b_{23} \cdots\cdots, \\ b_{13} & b_{23} & 1 \quad \cdots\cdots, \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$\therefore (1-\psi)^{-1} = 1/\Delta \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \cdots\cdots, \\ \beta_{21} & \beta_{22} & \beta_{23} \cdots\cdots, \\ \beta_{31} & \beta_{32} & \beta_{33} \cdots\cdots, \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

* Prof. Cayley, Philosophical Trans., 1858.

and

$$\therefore\ \phi=(1+\psi)(1-\psi)^{-1}=1/\Delta\begin{pmatrix} 1 & b_{12} & b_{13}\ldots\ldots, \\ -b_{12} & 1 & b_{23}\ldots\ldots, \\ -b_{13} & -b_{23} & 1\ \ldots\ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13}\ldots\ldots, \\ \beta_{21} & \beta_{22} & \beta_{23}\ldots\ldots, \\ \beta_{31} & \beta_{32} & \beta_{33}\ldots\ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

$$=1/\Delta\begin{pmatrix} \beta_{11}+b_{12}\beta_{21}+b_{13}\beta_{31} & \beta_{12}+b_{12}\beta_{22}+b_{13}\beta_{32} & \beta_{13}+b_{12}\beta_{23}+b_{13}\beta_{33}\ldots\ldots, \\ -b_{12}\beta_{11}+\beta_{21}+b_{23}\beta_{31} & -b_{12}\beta_{12}+\beta_{22}+b_{23}\beta_{32} & -b_{12}\beta_{13}+\beta_{23}+b_{23}\beta_{33}\ldots\ldots, \\ -b_{13}\beta_{11}-b_{23}\beta_{21}+\beta_{31} & -b_{13}\beta_{12}-b_{23}\beta_{22}+\beta_{32} & -b_{13}\beta_{13}-b_{23}\beta_{23}+\beta_{33}\ldots\ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

$$=\begin{pmatrix} 2\dfrac{\beta_{11}-\Delta}{\Delta} & 2\dfrac{\beta_{12}}{\Delta} & 2\dfrac{\beta_{13}}{\Delta} & \ldots\ldots, \\ 2\dfrac{\beta_{21}}{\Delta} & 2\dfrac{\beta_{22}-\Delta}{\Delta} & 2\dfrac{\beta_{23}}{\Delta} & \ldots\ldots, \\ 2\dfrac{\beta_{31}}{\Delta} & 2\dfrac{\beta_{32}}{\Delta} & 2\dfrac{\beta_{33}-\Delta}{\Delta} & \ldots\ldots, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

These representations fail for the case of an orthogonal matrix having both $+1$ and -1 as latent roots.

If we take

$$\phi_1=\theta\phi=\theta/\Delta\begin{pmatrix} 2\beta_{11}-\Delta & 2\beta_{12} & 2\beta_{13} & \ldots\ldots, \\ 2\beta_{21} & 2\beta_{22}-\Delta & 2\beta_{23} & \ldots\ldots, \\ 2\beta_{31} & 2\beta_{32} & 2\beta_{33}-\Delta & \ldots\ldots, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

where
$$\theta=\begin{pmatrix} b_{11} & 0 & 0\ \ldots\ldots, \\ 0 & b_{22} & 0\ \ldots\ldots, \\ 0 & 0 & b_{33}\ldots\ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}\quad\text{and}\quad \begin{array}{l} b_{11}=\pm 1, \\ b_{22}=\pm 1, \\ b_{33}=\pm 1, \\ \text{etc.} \end{array}$$

we will obviously have a matrix satisfying the conditions of orthogonality, and which includes ϕ as a special case.

Taking as an example a matrix of the 3[d] order, we have

$$
\begin{aligned}
|\phi_1+1| = b_{11}b_{22}b_{33} + \frac{1}{\Delta^2}\,&[b_{22}b_{33}\{(2\beta_{33}-\Delta)(2\beta_{22}-\Delta)-4\beta_{23}\beta_{32}\} \\
&+ b_{11}b_{22}\{(2\beta_{11}-\Delta)(2\beta_{22}-\Delta)-4\beta_{12}\beta_{21}\} \\
&+ b_{11}b_{33}\{(2\beta_{11}-\Delta)(2\beta_{33}-\Delta)-4\beta_{13}\beta_{31}\}] \\
+ \frac{1}{\Delta}\,&[b_{33}(2\beta_{33}-\Delta)+b_{22}(2\beta_{22}-\Delta)+b_{11}(2\beta_{11}-\Delta)] + 1 \\
= b_{11}b_{22}b_{33} + \frac{1}{\Delta^2}\,&[(b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33})(4\Delta+\Delta^2) - 2\Delta\{b_{22}b_{33}(\beta_{22}+\beta_{33}) \\
&+ b_{11}b_{22}(\beta_{11}+\beta_{22}) + b_{11}b_{33}(\beta_{11}+\beta_{33})\}] \\
+ \frac{1}{\Delta}\,&[2(b_{11}\beta_{11}+b_{22}\beta_{22}+b_{33}\beta_{33}) - \Delta(b \quad + b_{22}+b_{33})] + 1,
\end{aligned}
$$

and

$$
\begin{aligned}
|\phi_1-1| = b_{11}b_{22}b_{33} - \frac{1}{\Delta^2}\,&[(b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33})(4\Delta+\Delta^2) - 2\Delta\{b_{22}b_{33}(\beta_{22}+\beta_{33} \\
&+ b_{11}b_{22}(\beta_{11}+\beta_{22}) + b_{11}b_{33}(\beta_{11}+\beta_{33})\}] \\
+ \frac{1}{\Delta}\,&[2(b_{11}\beta_{11}+b_{22}\beta_{22}+b_{33}\beta_{33}) - \Delta(b_{11}+b_{22}+b_{33})] - 1.
\end{aligned}
$$

These results give

$$
\begin{aligned}
&|\phi_1+1| = 0 \text{ for } b_{11}=b_{22}=b_{33}=-1 \text{ or } b_{11}=-1 \text{ and } b_{22}=b_{33}=1,^* \\
&|\phi_1-1| = 0 \quad \text{"} \quad b_{11}=b_{22}=b_{33}=1 \quad \text{"} \quad b_{11}=1 \quad \text{"} \quad b_{22}=b_{33}=-1,^*
\end{aligned}
$$

$$
\begin{aligned}
|\phi_1+1| &= \frac{8b_{12}^2}{\Delta} \text{ for } b_{11}=b_{22}=-1 \text{ and } b_{33}=1, \\
&= \frac{8b_{13}^2}{\Delta} \quad \text{"} \quad b_{11}=b_{33}=-1 \text{ and } b_{22}=1, \\
&= \frac{8b_{23}^2}{\Delta} \quad \text{"} \quad b_{22}=b_{33}=-1 \text{ and } b_{11}=1,
\end{aligned}
$$

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* It is obvious that $|\phi_1+1|=0$ for $b_{22}=-1$ and $b_{11}=b_{33}=1$, and for $b_{33}=-1$ and $b_{11}=b_{22}=1$, as well as for the case given. Similarly in case of $|\phi_1-1|$.

and

$$|\phi_1 - 1| = -\frac{8b_{12}^2}{\Delta} \text{ for } b_{11} = b_{22} = \quad 1 \text{ and } b_{33} = -1,$$

$$= -\frac{8b_{13}^2}{\Delta} \text{ `` } b_{11} = b_{33} = \quad 1 \text{ and } b_{22} = -1,$$

$$= -\frac{8b_{23}^2}{\Delta} \text{ `` } b_{22} = b_{33} = \quad 1 \text{ and } b_{11} = -1.$$

The matrix ϕ_1 is therefore such that both $+1$ and -1 are latent roots, provided one of the constituents of ψ vanish.

Similarly for matrices of higher order.

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